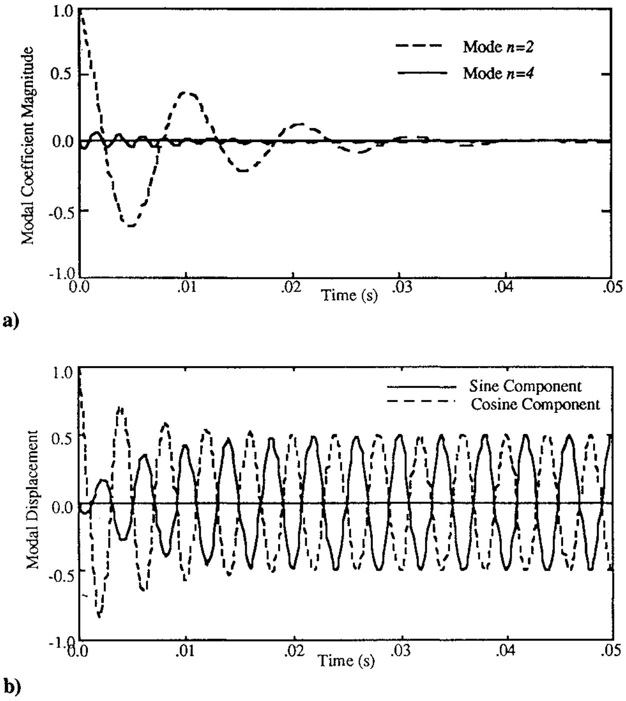

Fig. 1 Effect of control spillover ($M = 2, -4 \leq n \leq 4, \alpha = 100$): a) transient response of modes $n = 2$, and $n = 4$ and b) transient response of modes $n = -3$, and $n = 3$.

spaced nodes on the ring. Initially, the cosine part of mode $n = 3$ is excited. It then decays as the controller shifts some of its energy into the sine part. After about 0.015 s, the magnitudes of the sine part and the cosine part are about the same and there is little change from that point on. This shows that the control system has shifted the nodes of the real deformation pattern resulting from modes $n = \pm 3$ so that they lie very near the control forces. Hence, the control forces can no longer affect modes $n = \pm 3$. This phenomenon is referred to as mode rotation.

Conclusions

We have presented a method for uniform modal damping of the $2M + 1$ lowest modes of elastic rings by using velocity and position feedback of $2(M + 1)$ control forces evenly spaced around the ring. The method makes use of natural control and extends the node control theorem for uniform beams to uniform rings. The control system developed in this Note has been shown to provide uniform damping to the controlled modes of the ring and to preserve the mode shapes and natural frequencies of the controlled modes. The effects of control spillover have been studied in numerical simulations and the phenomena of mode rotation has been discussed.

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Alternative Variable Transformation for Simulation of Multibody Dynamic Systems

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I. Introduction

THE study of multibody dynamics (MBD) problems has emerged as an important area of research pertaining to the dynamics and control of spacecraft. The MBD system is usually modeled by a set of differential-algebraic equations, including a Jacobian matrix characterizing the kinematic constraints embedded in body-to-body connection. Based on such differential-algebraic equations, there exist numerous integration algorithms for open-loop and/or closed-loop dynamic simulations, including those given in Refs. 1 and 2. The algorithms in Refs. 1 and 2 are particularly simple because of the development of a modified null-space method using a natural partitioning scheme. This simplicity is made possible by determining a closed-form null-space matrix in an explicit way to nullify the Jacobian matrix so that the resulting equations of motion appear to be purely differential equations only in terms of independent variables. This approach has been proven very useful and effective in dealing with various types of the MBD systems.

For linear control design for slewing systems, Ghaemmaghami and Juang[3] and Huang et al.[4] derived an alternative state transformation matrix for the Lagrangian equation of motion to localize the nonlinearities, existing in the mass matrix, into a small submatrix associated with the rigid body. It has been shown in Refs. 3 and 4 that this characteristic leads to considerable savings in computer time during simulation and design phases. The purpose of the present Note is to introduce a new alternative variable transformation incorporating a null-space transformation for the MBD system, in which the mass matrix is transformed to a band matrix[5,6] which has many numerical advantages. Such a concept forms the basis of investigation in this Note and is also demonstrated as applied to an articulated MBD system.

II. Equations of Motion of an Articulated Multibody Dynamic System

We begin to consider an MBD system featuring an open-chain configuration of n bodies articulated by the spherical joints in the presence of external forces. The equations of motion of this articulated MBD system can be represented in matrix form as

$$\boldsymbol{M}\ddot{\boldsymbol{\xi}} + \boldsymbol{B}^T\lambda = \boldsymbol{F} \quad \text{and} \quad \boldsymbol{\Phi} = \boldsymbol{B}\dot{\boldsymbol{\xi}} = 0 \tag{1}$$

where $\boldsymbol{M}$ and $\boldsymbol{B}$ denote the mass and constraint Jacobian matrices, and λ, $\boldsymbol{F}$, and $\boldsymbol{\Phi}$ the Lagrange's multiplier, forcing, and nonholonomic constraint vectors, respectively. The variable $\ddot{\boldsymbol{\xi}}$ contains the linear and angular acceleration of each body in a vectorial fashion as $[\ddot{r}_1, \dot{\omega}_1, \ldots, \ddot{r}_n, \dot{\omega}_n]^T$, and the forcing vector includes the external forces as well as the control inputs when in a closed-loop system. In expanded form, the mass matrix can be written as follows:

$$\boldsymbol{M} = \text{Diag}[\boldsymbol{m}_1, \boldsymbol{j}_1, \ldots, \boldsymbol{m}_n, \boldsymbol{j}_n] \tag{2}$$

where $\boldsymbol{j}_i$ is a submatrix containing the moments of inertia for the ith body, and $\boldsymbol{m}_i$ is a component mass matrix.

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Moreover, the Jacobian matrix for this articulated multibody model can be written as

$$B = \begin{bmatrix} I & [\tilde{\ell}_{11}R_1]^T & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ I & [\tilde{\ell}_{12}R_1]^T & -I & -[\tilde{\ell}_{22}R_2]^T & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & [\tilde{\ell}_{23}R_2]^T & -I & -[\tilde{\ell}_{33}R_3]^T & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -I & -[\tilde{\ell}_{nn}R_n]^T \end{bmatrix} \tag{3}$$

where I is an identity matrix and R_i represents an Euler's rotational operator[1,2] associated with the ith body. In Eq. (3), $\tilde{\ell}_{ij}$ is formed by a skew-symmetric matrix in connection with a position vector directing from center of gravity (c.g.) of the ith body to the jth joint on the body-coordinate basis. Equation (1) thus represents a set of differential-algebraic equations for an articulated MBD system.

The Jacobian matrix B given by Eq. (3) can be nullified by an orthogonal complement, as a result of the use of a null-space method along with a natural partitioning scheme.[1,2] This in turn leads to the explicit determination of the equations of motion in the independent basis for an MBD system. This method is now applied to determine the complement of matrix B in a row-by-row expansion of the constraint vector Φ. Such a sequential procedure can be expressed as follows:

$$\dot{\xi} = A\dot{\xi}_\omega \quad \text{and} \quad \dot{\xi}_\omega = [\omega_1, \ldots, \omega_n]^T \tag{4}$$

where

$$A = \begin{bmatrix} -[\tilde{\ell}_{11}R_1]^T & 0 & 0 & \cdots & 0 \\ I & 0 & 0 & \cdots & 0 \\ R_1^T[\tilde{\ell}_{12} - \tilde{\ell}_{11}]^T & -[\tilde{\ell}_{22}R_2]^T & 0 & \cdots & 0 \\ 0 & I & 0 & \cdots & 0 \\ R_1^T[\tilde{\ell}_{12} - \tilde{\ell}_{11}]^T & R_2^T[\tilde{\ell}_{23} - \tilde{\ell}_{22}]^T & -[\tilde{\ell}_{33}R_3]^T & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R_1^T[\tilde{\ell}_{12} - \tilde{\ell}_{11}]^T & R_2^T[\tilde{\ell}_{23} - \tilde{\ell}_{22}]^T & R_3^T[\tilde{\ell}_{34} - \tilde{\ell}_{33}]^T & \cdots & -[\tilde{\ell}_{nn}R_n]^T \\ 0 & 0 & 0 & \cdots & I \end{bmatrix} \tag{5}$$

which is in fact a null-space matrix that satisfies an orthogonal relationship such as $BA = 0$. Note that the original variables (ξ) with redundancy in degree of freedom are transformed to the independent ones (ξ_ω) by means of this null-space transformation. Then, premultiplying Eq. (1) with the transpose of matrix A and substituting $\ddot{\xi}$ with $\dot{\xi}_\omega$ and $\ddot{\xi}_\omega$ derived from Eq. (4), one has

$$A^T M A \ddot{\xi}_\omega = A^T F - A^T M \dot{A} \dot{\xi}_\omega \tag{6}$$

Equation (6) thus provides a set of λ-free differential equations that are capable of implementing the dynamic simulation subjected to any external excitation or control input residing in the vector F.

It should be noted that the inversion of the matrix product $A^T MA$ in Eq. (6) may be chosen as an alternative means for the numerical integration, in which the acceleration variables $\ddot{\xi}_\omega$ are updated from the previous state within a time interval. Direct calculation of such an inversion is, however, very tedious, expensive, and time consuming, especially for cases where many time steps are required. Because of this, it is desirable to introduce an alternative variable transformation for the MBD system such that the matrix to be inverted is in a band format. This will be the subject in the next section.

III. Alternative Variable Transformation for a Multibody Dynamic System

In this section, an alternative variable transformation is considered in an attempt to arrange a band mass matrix for an articulated MBD system governed by Eq. (1). In the present approach, the following transformation of variables is first applied prior to the null-space transformation as described in Eq. (4):

$$\dot{\xi} = T\dot{\bar{\xi}} \quad \text{and} \quad \ddot{\xi} = T\ddot{\bar{\xi}} \tag{7}$$

where a linear transformation of variables from ξ to $\bar{\xi}$ is performed by a constant matrix T given below:

$$T = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & I & 0 & 0 & 0 & 0 & 0 & \cdots \\ I & 0 & I & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & I & 0 & 0 & 0 & \cdots \\ I & 0 & I & 0 & I & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & I & 0 & \cdots \\ I & 0 & I & 0 & I & 0 & I & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{8}$$

which is equivalent to the sum of a $3n \times 3n$ identity matrix and a lower-triangular matrix with many unity matrices of rank 3 placed as shown. The purpose of transformation matrix T is aimed at arranging the Jacobian matrix B and its complement A into a special format so that a band mass matrix can be obtained. Substituting Eq. (7) into the differential-algebraic equation (1) and premultiplying the resulting equation with the transpose of matrix T, one obtains

$$\bar{M}\ddot{\bar{\xi}} + \bar{B}^T \lambda = \bar{F} \quad \text{and} \quad \dot{\bar{\Phi}} = \bar{B}\dot{\bar{\xi}} = 0 \tag{9}$$

where the new matrices $\bar{M}$, $\bar{B}$, and $\bar{F}$ are formed by $T^T MT$, BT, and $T^T F$, respectively. Expansion of the new Jacobian matrix $\bar{B}$ provides

$$\bar{B} = \begin{bmatrix} I & [\tilde{\ell}_{11}R_1]^T & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & [\tilde{\ell}_{12}R_1]^T & -I & -[\tilde{\ell}_{22}R_2]^T & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & [\tilde{\ell}_{23}R_2]^T & -I & -[\tilde{\ell}_{33}R_3]^T & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -I & -[\tilde{\ell}_{nn}R]^T \end{bmatrix} \tag{10}$$

which is different from the one in Eq. (3) as it contains many subdiagonal unity matrices. Based on the new Jacobian matrix in Eq. (10), an orthogonal complement is derived in the same way as was presented in Eq. (4) and it can be expressed as

$$\dot{\bar{\xi}} = \bar{A}\dot{\bar{\xi}}_\omega \quad \text{and} \quad \ddot{\bar{\xi}} = \bar{A}\ddot{\bar{\xi}}_\omega + \dot{\bar{A}}\dot{\bar{\xi}}_\omega$$

$$\bar{A} = \begin{bmatrix} -[\tilde{\ell}_{11}R_1]^T & 0 & 0 & \cdots & 0 \\ I & 0 & 0 & \cdots & 0 \\ [\tilde{\ell}_{12}R_1^T]^T & -[\tilde{\ell}_{22}R_2]^T & 0 & \cdots & 0 \\ 0 & I & 0 & \cdots & 0 \\ 0 & [\tilde{\ell}_{23}R_2^T]^T & -[\tilde{\ell}_{33}R_3]^T & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -[\tilde{\ell}_{nn}R_n]^T \\ 0 & 0 & 0 & \cdots & I \end{bmatrix} \tag{11}$$

which in turn confirms the orthogonality, viz. $\bar{B}\bar{A} = 0$. As can be seen from Eq. (11), the above null-space transformation results in a band matrix $\bar{A}$ whose expression is simpler in appearance to that given in Eq. (5). Similar to Eq. (4), Eq. (11) is thus capable of nullifying the new Jacobian matrix $\bar{B}$ on an orthogonal basis while simultaneously generating a new set of independent variables ξ_ω after the null-space transformation. To eliminate the new Jacobian matrix $\bar{B}$ in Eq. (9), Eq. (9) is premultiplied with the transpose of matrix $\bar{A}$:

$$\bar{A}^T\bar{M}\bar{A}\ddot{\xi}_\omega = \bar{A}^T\bar{F} - \bar{A}^T\bar{M}\dot{\bar{A}}\dot{\xi}_\omega \tag{12}$$

which resembles the λ-free differential equation (6) derived in the previous section, except that an additional variable transformation is conducted herein. Unlike the matrix product A^TMA in Eq. (6), $\bar{A}^T\bar{M}\bar{A}$ appearing in Eq. (12) can be expressed in a closed form as

$$\bar{A}^T\bar{M}\bar{A} = \begin{bmatrix} \bar{m}_{11} & \bar{m}_{12} & 0 & 0 & \cdots & 0 & 0 \\ \bar{m}_{12}^T & \bar{m}_{22} & \bar{m}_{23} & 0 & \cdots & 0 & 0 \\ 0 & \bar{m}_{23}^T & \bar{m}_{33} & \bar{m}_{34} & \cdots & 0 & 0 \\ 0 & 0 & \bar{m}_{34}^T & \bar{m}_{44} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \bar{m}_{n-1,n-1} & \bar{m}_{n-1,n} \\ 0 & 0 & 0 & 0 & \cdots & \bar{m}_{n-1,n}^T & \bar{m}_{nn} \end{bmatrix} \tag{13}$$

where the submatrices $\bar{m}_{ij}$ can be described with the aid of indicial notation given below:

On-Diagonal:

$$\bar{m}_{ii} = \bar{J}_i + \sum_{k=i}^{i+1}[\tilde{\ell}_{1k}R_1]\bar{m}_k[\tilde{\ell}_{1k}R_1]^T, \quad \text{for } i = 1,\ldots,n-1$$

$$\bar{J}_n + [\tilde{\ell}_{nn}R_n]\bar{m}_n[\tilde{\ell}_{nn}R_n]^T, \quad \text{for } i = n$$

Off-Diagonal:

$$\bar{m}_{ij} = -[\tilde{\ell}_{ij}R_i]\bar{m}_j[\tilde{\ell}_{jj}R_j]^T, \quad \text{for } \begin{matrix} i = 1,\ldots,n-1 \text{ and} \\ j = i+1 \end{matrix} \tag{14}$$

Due to the simple format of the band matrix, an inverse successive displacement scheme[5,6] can be applied to determine ξ_ω in integrating Eq. (12) for simulation, which only demands a successive inversion of the 3×3 submatrices on the diagonal. As a result, only $6n$ matrix operations are needed for the determination of the cofactors in this successive inverse process, while $\frac{3}{2}n(3n+1)$ operations are deemed necessary for the inversion of a $3n \times 3n$ matrix given by Eq. (6). As such, significant time saving is achievable for the simulation of the MBD system in conjunction with the use of the band mass matrix resulting from an alternative variable transformation. Once the updated solution of ξ_ω is obtained from Eq. (12), the original acceleration variables can then be determined through the use of Eqs. (7) and (11).

IV. Conclusions

An alternative variable transformation has been proposed and analyzed for the simulation of multibody dynamic systems. The alternative variable transformation can be used by incorporating a modified null-space method to transform the mass matrix of multibody dynamic equations into a closed-form band fashion, which results in an efficient computation for the acceleration variables during integration. An alternative variable transformation matrix has been developed to work directly with the multibody equations of motion without altering the inherent dynamic characteristics, and it eliminates the need for expensive computation of inversion of a large mass matrix which is required for the simulation of multibody dynamic systems. An articulated multibody model has been selected for an analytical derivation, in which it has shown that the computation can be saved $(3n + 1/4)$-fold due to the computational merits associated with the band matrix.

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Trajectory Optimization Using Parallel Shooting Method on Parallel Computer

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Introduction

TRAJECTORY optimization problems arise frequently in the design of modern guidance and control systems. A variety of techniques are currently available for the solution of trajectory optimization problems.[1–3] They include shooting methods, gradient-based methods, and methods using nonlinear programming. This Note considers the family of shooting methods that are used when a high-accuracy solution is required. In shooting methods, the unknown conditions are estimated, and the differential equations are integrated. The initial estimates are then iteratively corrected using Newton's method.

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